Weyl approach to representation theory of reflection equation algebra

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Abstract

The present paper deals with the representation theory of the reflection equation algebra, connected with a Hecke type R-matrix. Up to some reasonable additional conditions the R-matrix is arbitrary (not necessary originated from quantum groups). We suggest a universal method of constructing finite dimensional irreducible non-commutative representations in the framework of the Weyl approach well known in the representation theory of classical Lie groups and algebras. With this method a series of irreducible modules is constructed which are parametrized by Young diagrams. The spectrum of central elements $s_k = Tr_q\, L^k$ is calculated in the single-row and single-column representations. A rule for the decomposition of the tensor product of modules into the direct sum of irreducible components is also suggested.

1 Reflection Equation Algebra

Reflection equation and the corresponding algebra which will be called the reflection equation algebra (REA for short) play a significant role in the theory of integrable systems and non-commutative geometry. In application to integrable systems the reflection equation with a spectral parameter is mainly used. First it appears in the work by I. Cherednik [1]. Usually it comprises the information about the behaviour of a system at a boundary, for example, describes the reflection of particles on a boundary of the configuration space.

The reflection equation without a spectral parameter is important for the non-commutative geometry. One of the first applications the corresponding REA found in the theory of differential calculus on quantum groups (see, e.g., [2]). In such a differential calculus REA with the Hecke type R-matrix is a non-commutative analog of the algebra of vector fields on the groups $GL(N)$ or $SL(N)$. Besides, REA serves as a base for a definition of quantum analogs of homogeneous spaces — orbits of the coadjoint representation of a Lie group, as well as quantum analogs of linear bundles over such orbits (see, e.g., [3, 4]).

In this paper we turn to problems of the representation theory of REA without a spectral parameter. We are interested in the following main topics:

i) a construction of finite dimensional non-commutative irreducible representations and the calculation of spectrum (characters) of central elements in these representations;

ii) a rule for the decomposition of the tensor product of irreducible modules into irreducible components.

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Before reviewing the known results, we introduce some necessary definitions and notations.

Consider an associative algebra $\mathcal{L}_q$ with the unity $e_{\mathcal{L}}$ over the complex field $\mathbb{C}$ generated by n^2 elements $\hat{l}_i^{\,j}$, $1 \le i, j \le n$, n being a fixed positive integer. Let the generators satisfy the following quadratic commutation relations

$$R_{12}\hat{L}_1 R_{12}\hat{L}_1 - \hat{L}_1 R_{12}\hat{L}_1 R_{12} = 0, \quad \hat{L}_1 \equiv \hat{L} \otimes I, \tag{1.1}$$

where the matrix $\hat{L} \in \mathrm{Mat}_n(\mathcal{L}_q)$ is composed of $\hat{l}_i^{\,j}$: $\hat{L} = \|\hat{l}_i^{\,j}\|$. Here the lower index enumerates the rows while the upper one columns. In (1.1) and everywhere below the use is made of the compact matrix notations [5] when the index of an object indicates the vector space to which the object belongs (or in which this object acts). The symbol I stands for the unity matrix whose dimension is always clear from the context of formulae. A numerical $n^2 \times n^2$ matrix R is a solution of *the Yang-Baxter equation*

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{1.2}$$

The algebra $\mathcal{L}_q$ described above will be called *the reflection equation algebra* (REA).

Impose now several additional conditions on the matrix R. First of them is *the Hecke condition*

$$(R - qI)(R + q^{-1}I) = 0. \tag{1.3}$$

The parameter q is a fixed nonzero complex number with the only constraint[1]

$$q^k \ne 1, \quad \forall\, k \in \mathbb{N}. \tag{1.4}$$

As a consequence, the q-analogs of all integers are nonzero

$$k_q \equiv \frac{q^k - q^{-k}}{q - q^{-1}} \ne 0, \quad \forall\, k \in \mathbb{N}. \tag{1.5}$$

Besides, we shall suppose the R-matrix to be *skew-invertible* that is there exists an $n^2 \times n^2$ matrix Ψ such that

$$\sum_{a,b} R_{ia}^{jb}\Psi_{bk}^{as} = \delta_i^s \delta_k^j = \sum_{a,b} \Psi_{ia}^{jb} R_{bk}^{as}.$$

In the compact notations the above formula reads

$$Tr_{(2)} R_{12}\Psi_{23} = P_{13} = Tr_{(2)}\Psi_{12}R_{23}, \tag{1.6}$$

where the symbol $Tr_{(2)}$ means the calculation of trace in the second space and P is the permutation matrix.

To formulate the last requirement on R one should consider the connection of the A_k series Hecke algebras with the group algebras of finite symmetric groups. One of the simplest definition of the Hecke algebra reads as follows.

Fix a nonzero complex number q. The *Hecke algebra of A_k series* ($k \ge 2$) is an associative algebra $H_k(q)$ over the complex field $\mathbb{C}$ generated by the unit element 1_H and $k-1$ generators σ_i subject to the following relations:

$$\left.\begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \ge 2 \\ (\sigma_i - q\,1_H)(\sigma_i + q^{-1}\,1_H) = 0 & \end{array}\right\} \quad i = 1, 2, \ldots k-1.$$

[1] All our subsequent constructions possess a well defined "classical limit" $q \to 1$. This limit corresponds to REA with an involutive R-matrix: $R^2 = I$.

In some cases it proves to be convenient to consider q as a formal parameter and consider the Hecke algebra over the field of rational functions in the indeterminate q. We shall always bear in mind this extension when considering the classical limit $q \to 1$.

Let us treat R as the matrix of a linear operator (in a fixed basis) which acts in the tensor square $V^{\otimes 2}$ of a finite dimensional vector space V, $\dim V = n$. Then an arbitrary Hecke R-matrix define *the local representation* of $H_k(q)$ in $V^{\otimes k}$

$$\sigma_i \to \rho_R(\sigma_i) = R_{ii+1} = I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(k-i-1)} \in \mathrm{End}(V^{\otimes k}). \tag{1.7}$$

If the parameter q satisfies (1.4), then for any positive integer k the Hecke algebra $H_k(q)$ is known to be isomorphic to the group algebra $\mathbb{C}[\mathcal{S}_k]$ of the k-th order permutation group $\mathcal{S}_k$. As a consequence, there exist elements $\mathcal{Y}_{\nu(a)}(\sigma) \in H_k(q)$ which are the q-analogs of the Young idempotents (projectors) widely used in the theory of symmetric groups. These q-idempotents are parametrized by standard Young tableaux $\nu(a)$ corresponding to each diagram or equivalently to each partition $\nu \vdash k$. The number of all standard tableaux $\nu(a)$ which one can construct for a given ν will be denoted $\dim[\nu]$

$$\dim[\nu] = \#\{\nu(a)\}.$$

In the local representation (1.7) the elements $\mathcal{Y}_\nu(\sigma)$ are realized as some projector operators in $V^{\otimes k}$. With respect to the action of these projectors the space $V^{\otimes k}$ is decomposed into the direct sum of subspaces V_ν as in the case of the symmetric group

$$V^{\otimes k} = \bigoplus_{\nu \vdash k} \bigoplus_{a=1}^{\dim[\nu]} V_{\nu(a)}, \quad V_{\nu(a)} = Y_{\nu(a)}(R) \triangleright V^{\otimes k}. \tag{1.8}$$

The projector $Y_{\nu(a)}(R) = \rho_R(\mathcal{Y}_{\nu(a)})$ is given by some polynomial in matrices R_{ii+1}. For detailed treatment of these questions, explicit formulae for q-projectors and the extensive list of original papers the reader is referred to [6].

So, we shall assume that there exists an integer $p > 0$ such that the image of the q-antisymmetrizer $\mathcal{A}^{(p+1)}(\sigma) \in H_k(q)$ $(\forall\, k > p)$ under the local R-matrix representation ρ_R is identical zero while the image of the q-antisymmetrizer $\mathcal{A}^{(p)}(\sigma) \in H_k(q)$ is a unit rank projector in the space $V^{\otimes k}$

$$\exists\, p \in \mathbb{N}: \quad \begin{cases} \mathcal{A}^{(p+1)}(\sigma) \xrightarrow{\rho_R} A^{(p+1)}(R) \equiv 0, \\ \mathcal{A}^{(p)}(\sigma) \xrightarrow{\rho_R} A^{(p)}(R), \quad \mathrm{rank}\, A^{(p)}(R) = 1. \end{cases} \tag{1.9}$$

Such a number p will be called *the symmetry rank* of the matrix R. For example, the symmetry rank of R-matrix connected with the quantum universal enveloping algebra $U_q(sl_n)$ is equal to n. Examples of $n^2 \times n^2$ R-matrices with $p < n$ (for $n \geq 3$) were found in [7].

Introduce now two $n \times n$ matrices B and C

$$B_1 = Tr_{(2)}\Psi_{21}, \quad C_1 = Tr_{(2)}\Psi_{12}, \tag{1.10}$$

where Ψ is defined in (1.6). If the R-matrix has the symmetry rank p these matrices are nonsingular and their product is a multiple of the unit matrix [7]

$$B \cdot C = \frac{1}{q^{2p}}\, I. \tag{1.11}$$

Besides, B and C have the following traces

$$Tr B = Tr C = \frac{p_q}{q^p}. \tag{1.12}$$

The matrices B and C play the central role in what follows.

The simplest example of REA is obtained by choosing the $U_q(sl_2)$ R-matrix $(n=2)$

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad \lambda \equiv q - q^{-1}, \qquad \hat{L} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}.$$

In this case equation (1.1) leads to six permutation relations for the generators of REA

$$\begin{array}{rl} q^2\hat{a}\hat{b} = \hat{b}\hat{a} & \qquad q(\hat{b}\hat{c} - \hat{c}\hat{b}) = \lambda\,\hat{a}(\hat{d} - \hat{a}) \\ q^2\hat{c}\hat{a} = \hat{a}\hat{c} & \qquad q(\hat{c}\hat{d} - \hat{d}\hat{c}) = \lambda\,\hat{c}\hat{a} \\ \hat{a}\hat{d} = \hat{d}\hat{a} & \qquad q(\hat{d}\hat{b} - \hat{b}\hat{d}) = \lambda\,\hat{a}\hat{b}. \end{array} \tag{1.13}$$

Consider a map $Tr_q : \mathrm{Mat}_n(\mathcal{L}_q) \to \mathcal{L}_q$ which is called *the quantum trace* [5]

$$Tr_q(X) \stackrel{\mathrm{def}}{=} Tr(C \cdot X), \quad X \in \mathrm{Mat}_n(\mathcal{L}_q). \tag{1.14}$$

One can show that the quantities

$$s_m(\hat{L}) = Tr_q(\,\hat{L}^m) \quad 1 \le m \le p-1 \tag{1.15}$$

are *independent* central elements of REA (see [5]). Presumably these elements (together with $e_{\mathcal{L}}$) generate the whole center of REA, but we do not know the proof of this hypothesis. Calculation of spectrum of central elements s_m in irreducible representations of $\mathcal{L}_q$ is one of our aims.

At present there exist rather lot of works, devoted to the representation theory of REA. First of all, this algebra possesses a large number of one-dimensional (commutative) representations. For example, it is evident that at any choice of R-matrix relation (1.1) will be satisfied if one sets $\hat{l}_i^{\,j} = \alpha\,\delta_i^j$. Less trivial representation can be obtained for our simple example (1.13) by putting $\hat{a} = 0$. Then the remaining generators are represented by three arbitrary complex numbers. Such like representations were considered in detail in [8] for the REA with R-matrices coming from $U_q(sl_n)$ and its super-symmetric generalizations.

But since REA is a non-commutative algebra, the images of a part of its generators are inevitably zero in any one-dimensional representation. That is the kernel of any one-dimensional representation of REA must contain some of its generators. These representations are not comprised by our approach and we shall not consider them.

The main object of our interest will be the non-commutative representations which for *any* generator of REA put into correspondence a nontrivial linear operator in a finite dimensional vector space. An example of such a representation can be constructed in the following way. Let us use the fact that REA (1.1) is an adjoint comodule over some Hopf algebra which is similar to to the algebra of functions over the quantum group. Suppose the commutations among the generators t_i^j of the Hopf algebra to be given by the matrix relation [5]

$$R_{12}T_1T_2 = T_1T_2R_{12}. \tag{1.16}$$

The above multiplication is compatible with the comultiplication Δ

$$t_i^j \stackrel{\Delta}{\longrightarrow} \sum_k t_i^k \otimes t_k^j. \tag{1.17}$$

Then (1.1) is covariant with respect to the transformation [9]

$$\hat{l}_i^j \to t_i^k S(t_p^j) \otimes \hat{l}_k^p, \tag{1.18}$$

where $S(\cdot)$ stands for the antipodal map[2]. So, if one knows the representations of Hopf algebra (1.16) then, given a representation of REA, one can construct another one on the base of (1.18).

Moreover, in the case of $U_q(sl_n)$ R-matrix one has an additional possibility of constructing non-commutative representations. The matter is that in such a case there exists an embedding of REA into $U_q(gl_n)$. At the level of generators this embedding is described by the formula [5]

$$\hat{L} = S(L^-)L^+, \tag{1.19}$$

where $L^{\pm}$ are matrices composed of the $U_q(gl_n)$ generators. Therefore, starting from a representation of the quantum group one can find the corresponding representation of REA by means of (1.19). In recent paper [10] this approach was extended to the case of an arbitrary quasitriangular Hopf algebra. The authors of the cited paper constructed a universal solution of (1.1) basing on the universal R-matrix of a quasitriangular Hopf algebra $\mathcal{H}$. The generators $\hat{l}_i^j$ turn out to be elements of the tensor product of $\mathcal{H}$ and its "twisted dual" algebra.

However, it is worth pointing out that all the methods mentioned above are essentially based on the representation theory of objects which are external to REA, namely, on the theory of quasitriangular Hopf algebras, the quantum groups being a particular case of them. But as was shown in [11], the Yang-Baxter equation possesses a lot of solutions do not connected with a quantum group (see, also, [7]). For such type solutions we cannot use map (1.19) and cannot construct the REA representations on the base of quantum group ones.

Moreover, having fixed the R-matrix in (1.1), one completely defines all properties of REA and its representation theory as well. The situation is similar to the Lie algebra theory, where the set of structure constants defines all properties of the algebra. Therefore, it is quite natural to develop the representation theory of REA using only the given R-matrix, that is entirely in terms of REA itself.

The most efficient method to solve this problem seems to consist in the direct analysis of the explicit commutation relations of the REA generators. For $U_q(sl_n)$ R-matrix (at small values of n) one can proceed in an analogy with the representation theory of the universal enveloping algebra of a (simple) Lie algebra. Following this way, P.P. Kulish [12] succeeded in finding all highest vector representations of the simplest REA (1.13). Besides two one-dimensional representations this algebra has a series of finite dimensional irreducible non-commutative representations and one infinite dimensional representation — an analog of the Verma module of the universal enveloping algebra.

Unfortunately, this approach is not universal. It is in essential dependence on the particular choice of R-matrix. The explicit components of matrix relations (1.1) may become completely different when we change the R-matrix. Therefore, one would have to repeat the analysis of commutation relations from the very beginning for each possible R-matrix. Another obstacle in this way is more technical. The matter is that even in the case of $U_q(sl_n)$ R-matrix the complexity of the explicit form of (1.1) increases very quickly with growing of n. This leads to additional difficulties as compared with the case of universal enveloping algebra, when the commutation relations among generators can be written in a compact form for an arbitrary n.

[2]Note, that it is skew-invertibility (1.6) which allows one to define the antipode in (bi)algebra (1.16) (to be more precise, in some its extension, see [5]).

In the present paper we suggest a universal method of constructing finite dimensional representations of REA generated by (1.1). These representations are parametrized by Young diagrams and exist for *any* R-matrix satisfying the additional conditions (1.3), (1.6) and (1.9). For the representations corresponding to single-row and single-column diagrams we calculate the spectrum of central elements (1.15). In the particular case of $U_q(sl_2)$ R-matrix our result reproduces the series of finite dimensional representations of REA (1.13) obtained in [12].

The paper is organized as follows. In Section 2 we construct an irreducible representation of REA with an arbitrary Hecke R-matrix possessing a finite symmetry rank. This representation is called the fundamental one (of B type) since its tensor products are decomposed into irreducible components similarly to those of fundamental vector representation of $U(gl_n)$.

In Section 3 we study the k-th tensor power of the fundamental module of B type and consider its decomposition into higher dimensional REA modules.

Section 4 is devoted to another fundamental module (of R type). The construction is based on the general theory of dual Hopf algebras and can be easily generalized to the case of an arbitrary Hecke R-matrix. The connection of B and R type fundamental modules is established. We also consider an example of reducible indecomposable module over REA which is not equivalent to either B or R type module. At the end of the section we give a short resumé of the obtained results and mention some open questions of the suggested approach.

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2 Fundamental module of B type

Consider the REA generated by relations (1.1) and make the linear shift of generators

$$l_i^j = \hat{l}_i^j + \frac{1}{\lambda}\delta_i^j\, e_{\mathcal{L}}, \tag{2.1}$$

where $e_{\mathcal{L}}$ is the unit element of $\mathcal{L}_q$ and $\lambda = q - q^{-1}$. On taking into account the Hecke condition (1.3) one obtains the commutation relations for the new generators

$$R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12} = R_{12}L_1 - L_1R_{12}. \tag{2.2}$$

In what follows we shall call the algebra generated by (2.2) *the modified reflection equation algebra* (mREA) and retain the notation $\mathcal{L}_q$ for it. Note, that unless $q = 1$ mREA is isomorphic to REA with relations (1.1). As a consequence, any representation of mREA can be transformed into that of REA and vice versa. Nevertheless, these algebras are different at the classical limit since isomorphism (2.1) is broken at $q \to 1$ in virtue of singularity of λ^{-1}.

In the particular case of $U_q(sl_n)$ R-matrix the classical limit of (2.2) gives the commutation relations of the $U(gl_n)$ generators.

So, consider the mREA $\mathcal{L}_q$ generated by the unit element and n^2 generators l_i^j with commutation relations (2.2). Let us take an n-dimensional vector space V and fix an arbitrary basis of n vectors

e_i, $1 \le i \le n$. Define a *linear* map $\pi: \mathcal{L}_q \to \mathrm{End}(V)$ in accordance with the rules

$$
\begin{aligned}
&\pi(e_{\mathcal{L}}) = \mathrm{id}_V \\
&\pi(l_i^{\,j}) \triangleright e_k = e_i B_k^{\,j}, \\
&\pi(l_1 \cdot l_2 \cdot \ldots \cdot l_k) = \pi(l_1) \cdot \pi(l_2) \cdot \ldots \cdot \pi(l_k), \quad \forall\, k \in \mathbb{N},
\end{aligned}
\tag{2.3}
$$

where the matrix B is defined in (1.10) and id_V is the identical operator on V.

Proposition 1 ([15]) *The linear map (2.3) defines an irreducible representation of $\mathcal{L}_q$ (2.2) in the space V. This will be called the fundamental module of B type.*

Proof To prove that π realizes the representation of $\mathcal{L}_q$ one should only verify that operators (2.3) do satisfy (2.2). This can be easily done by a straightforward calculation. The only fact needed in this way consists in the following simple consequence of (1.6) and (1.10)

$$
Tr_{(1)} B_1 R_{12} = I. \tag{2.4}
$$

Irreducibility follows from the non-singularity of B (1.11). Using this fact one can show that the operators $\pi(l_i^{\,j})$ span $\mathrm{End}(V)$ and, therefore, the space V does not contain proper invariant subspaces with respect to π. ■

If we make shift (2.1) in our example (1.13) we get the commutation relations of the corresponding mREA

$$
\begin{array}{ll}
q^2 ab - ba = qb & \qquad q(bc - cb) = (\lambda a - 1)(d - a) \\
q^2 ca - ac = qc & \qquad q(cd - dc) = c(\lambda a - 1) \\
ad = da & \qquad q(db - bd) = (\lambda a - 1)b.
\end{array}
\tag{2.5}
$$

Here in the right column the unit stands for $e_{\mathcal{L}}$. The matrices B and C have the form

$$
B = \begin{pmatrix} q^{-1} & 0 \\ 0 & q^{-3} \end{pmatrix}, \qquad C = \begin{pmatrix} q^{-3} & 0 \\ 0 & q^{-1} \end{pmatrix}.
$$

The fundamental representation (2.3) reads

$$
\pi(a) = \begin{pmatrix} q^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \; \pi(b) = \begin{pmatrix} 0 & q^{-3} \\ 0 & 0 \end{pmatrix}, \; \pi(c) = \begin{pmatrix} 0 & 0 \\ q^{-1} & 0 \end{pmatrix}, \; \pi(d) = \begin{pmatrix} 0 & 0 \\ 0 & q^{-3} \end{pmatrix}. \tag{2.6}
$$

Given a representation of $\mathcal{L}_q$ one can find the corresponding representation of the quotient algebra

$$
\mathcal{SL}_q = \mathcal{L}_q / \{Tr_q L\}, \tag{2.7}
$$

where $\{X\}$ stands for the ideal generated by a given subset $X \subset \mathcal{L}_q$. The commutation relations among the generators $f_i^{\,j}$ of $\mathcal{SL}_q$ has the same form (2.2) as those of $\mathcal{L}_q$ (with substitution $L \to F$ where $F = \|f_i^{\,j}\|$), but now the generators are linear dependent due to $Tr_q F = 0$.

At $q \to 1$ in the case of $U_q(sl_n)$ R-matrix the commutation relations of the $\mathcal{SL}_q$ generators transform into those of $U(sl_n)$ generators. For this reason the passage from $\mathcal{L}_q$ to $\mathcal{SL}_q$ (or from the $\mathcal{L}_q$ representation to the corresponding $\mathcal{SL}_q$ one) will be loosely called the sl-reduction in what follows.

The transformation of an irreducible $\mathcal{L}_q$ representation ρ acting in a finite dimensional space V into the $\mathcal{SL}_q$ representation $\bar\rho$ is realized as follows. Due to Tr_qL is a central element of $\mathcal{L}_q$ and ρ is an irreducible representation one gets

$$\rho(Tr_qL) = \chi(Tr_qL)\,\mathrm{id}_V \equiv \chi_1\,\mathrm{id}_V,$$

where $\chi : Z(\mathcal{L}_q) \to \mathbb{C}$ is a character of the center $Z(\mathcal{L}_q)$. Then the straightforward calculation shows that the $\mathcal{SL}_q$ generators $f_i^{\,j}$ in representation $\bar\rho$ are given by

$$\bar\rho(f_i^{\,j}) = \frac{1}{\omega}\Bigl(\rho(l_i^{\,j}) - \delta_i^j\,\frac{\chi_1}{TrC}\,\mathrm{id}_V\Bigr), \qquad \omega = 1 - \lambda\,\frac{\chi_1}{TrC}. \tag{2.8}$$

The traceless property $\bar\rho(Tr_qF) = 0$ is evident and the factor ω^{-1} ensures the correct normalization of the right hand side of (2.2).

Remark 1 *As can be easily seen from definition (1.1) the REA admits the "renormalization" automorphism $\hat l_i^{\,j} \to z\hat l_i^{\,j}$ with nonzero complex number z. The same is true for the REA representations as well. At the level of mREA representations this automorphism reads*

$$\rho(l_i^{\,j}) \to \rho_z(l_i^{\,j}) = z\rho(l_i^{\,j}) + \delta_i^j\,\frac{1-z}{\lambda}\,\mathrm{id}_V, \tag{2.9}$$

where ρ is an arbitrary mREA representation in the space V. Basing on (2.8) one can show that the corresponding $\mathcal{SL}_q$ representation $\bar\rho$ does not depend on z that is the whole class of mREA representations ρ_z connected by renormalization automorphism (2.9) gives the same $\mathcal{SL}_q$ representation $\bar\rho$.

Let us now obtain the sl-reduction of the B type representation π defined by (2.3). In virtue of (1.11) one finds

$$\chi_1 = \chi(Tr_qL) = q^{-2p}.$$

Then, taking into account (1.12) and (2.8) we find the B type representation $\bar\pi$ of algebra (2.7)

$$\bar\pi(f_i^{\,j}) = \frac{1}{\omega}\Bigl(\pi(l_i^{\,j}) - \frac{\delta_i^j}{q^p p_q}\,\mathrm{id}_V\Bigr), \quad \omega = \frac{q^{1-p}}{p_q}\,(q^{p-2}(p+1)_q - 1). \tag{2.10}$$

Consider again our example (2.5) of mREA $\mathcal{L}_q$. To get the corresponding algebra $\mathcal{SL}_q$ it is necessary to take the quotient of $\mathcal{L}_q$ over the ideal generated by Tr_qL. Using the explicit form of C one has

$$Tr_qL = \frac{1}{q^3}\,a + \frac{1}{q}\,d.$$

With a new generator $h = a - d$ one can rewrite the commutation relations for $\mathcal{SL}_q$ in terms of three independent quantities b, c and h

$$\begin{aligned} &q^2hb - bh = 2_q b\\ &hc - q^2ch = -2_q c\\ &q(bc - cb) = h(1 - \frac{q\lambda}{2_q}\,h). \end{aligned} \tag{2.11}$$

Note, that at $q \to 1$ relations (2.11) transforms into the well known commutation relations for the generators of the universal enveloping algebra $U(sl_2)$.

Starting from (2.6) we have due to (2.10)

$$\bar\pi(h)=\xi\begin{pmatrix} q & 0\\ 0 & -q^{-1}\end{pmatrix},\quad \bar\pi(b)=\xi\begin{pmatrix} 0 & q^{-1}\\ 0 & 0\end{pmatrix},\quad \bar\pi(c)=\xi\begin{pmatrix} 0 & 0\\ q & 0\end{pmatrix},\qquad \xi=\frac{q^2+1}{q^4+1}.$$

At the classical limit this representation turns into the fundamental vector representation of $U(sl_2)$.

Let us point out that at $q\neq\pm1$ the usual trace of $\bar\pi(h)$ does not equal to zero (this property is restored at the classical limit only). However, the quantum trace of the matrix $\bar\pi(h)$ is equal to zero

$$Tr(C\cdot\bar\pi(h))=0.$$

It should be emphasized that in the above relation the matrix C is used to deform the trace of *operators* of $\mathcal{SL}_q$ representation, while in (1.14) the quantum trace is taken in $\mathrm{Mat}_n(\mathcal{SL}_q)$ (or in $\mathrm{Mat}_n(\mathcal{L}_q)$).

So, in order to retain the traceless property of $\mathcal{SL}_q$ representation in the space V one has to modify the definition of the operator trace in $\mathrm{End}(V)$: $Tr\to Tr_q$. This fact is no mere chance. The usual tensor category like that of modules over $U(sl_n)$ is not suitable as the representation category for the algebra $\mathcal{L}_q$ (or $\mathcal{SL}_q$). The natural representation category for the mentioned algebras is some *quasitensor*[3] *Schur-Weyl category*. It is the quantum trace (contrary to the usual one) which turns out to be a natural morphism closely connected with the structure of the Schur-Weyl category. The detailed description of the category, the role of the quantum trace in it and the connection with REA are considered in [14] and [15].

To complete the section, we clarify the connection of B type representation (2.6) of mREA (2.5) with the result of [12]. Applying the inverse linear shift of generators (see (2.1)) to representation (2.6) one gets the representation of REA with quadratic relations (1.13). The matrices thus obtained are equivalent (connected by a similarity transform) to the matrices of the two-dimensional representation derived in [12]. This representation is the lowest one in the series of non-commutative finite dimensional representations of (1.13). The higher dimensional modules of this series are equivalent to the q-symmetrical tensor powers of the B type modules. Their construction is considered in the next section. So, in the particular case of $U_q(sl_2)$ R-matrix our approach gives the known result of [12].

3 Higher dimensional modules of B type

Let us turn to the problem of tensor product of mREA modules. We are mainly interested in the following questions. Firstly, given the fundamental module V of B type, how to define an mREA module structure in the tensor power $V^{\otimes k}$? Secondly, into which irreducible higher dimensional modules one can decompose the module $V^{\otimes k}$? At last, how one can decompose the tensor product of arbitrary higher dimensional modules (not only fundamental ones) into the sum of irreducible components? Here we propose answers to these questions.

3.1 Tensor product of fundamental modules

When trying to define the mREA module structure on the tensor product of fundamental modules one finds a serious difficulty. The matter is that it is not known if mREA (2.2) (as well as REA

[3]The notion of the quasitensor category as a category of finite dimensional representations of a quasitriangular Hopf algebra was introduced in [13].

(1.1)) possesses the *bialgebra* structure. As a consequence, in algebra (2.2) one cannot define the coproduct operation.

To clarify the importance of this operation, consider the case of the universal enveloping algebra $\mathcal{U} = U(sl_n)$ and dwell upon the definition of the $\mathcal{U}$-module structure on the tensor product of fundamental modules in the framework of Weyl approach [16].

The algebra $\mathcal{U}$ is a bialgebra[4] with the cocommutative coproduct

$$\Delta : \ \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}.$$

The action of Δ on a (Lie) generator[5] x of $\mathcal{U}$ is given by a simple formula

$$\Delta(x) = x \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes x \stackrel{\rm def}{=} x_{(1)} + x_{(2)},$$

where $1_{\mathcal{U}}$ is the unit element of $\mathcal{U}$.

Take now some irreducible representation $\rho : \mathcal{U} \to \mathrm{End}(V)$ of $\mathcal{U}$ in a finite dimensional vector space V. To get a representation of $\mathcal{U}$ in $V^{\otimes k}$ one first construct a homomorphism $\Delta^k : \mathcal{U} \to \mathcal{U}^{\otimes k}$ by multiple application of the coproduct Δ. Then the image of $\mathcal{U}$ under such a homomorphism is represented in $V^{\otimes k}$ with the help of the map $\rho^{\otimes k}$. For a given generator x of $\mathcal{U}$ these two steps can be written in the explicit form

$$\begin{array}{ll} i) & x \xrightarrow{\Delta^k} \mathbf{x} = x_{(1)} + x_{(2)} + \ldots + x_{(k)} \in \mathcal{U}^{\otimes k} \\ ii) & \mathbf{x} \to \rho^{\otimes k}(\mathbf{x}) \in \mathrm{End}(V^{\otimes k}). \end{array} \tag{3.1}$$

This representation is reducible. To extract the irreducible components one uses the fact that in $V^{\otimes k}$ it is possible to define a natural representation of the group algebra $\mathbb{C}[\mathcal{S}_k]$ of the k-th order permutation group $\mathcal{S}_k$. With respect to this representation the space $V^{\otimes k}$ is decomposed into the direct sum of irreducible $\mathbb{C}[\mathcal{S}_k]$ modules $V_{\nu(a)}$. The modules are parametrized by the standard Young tableaux $\nu(a)$, corresponding to all possible partitions $\nu \vdash k$. With respect to representation of $\mathcal{U}$ the subspaces $V_{\nu(a)}$ are also irreducible. A generator x of $\mathcal{U}$ is represented in $V_{\nu(a)}$ by the following linear operator

$$\rho_{\nu(a)}(x) = P_{\nu(a)}\, \rho^{\otimes k}(\mathbf{x})\, P_{\nu(a)}, \tag{3.2}$$

where $P_{\nu(a)}$ is the Young projector in $V^{\otimes k}$ corresponding to the tableau $\nu(a)$. The modules parametrized by different tableaux of the same partition ν are equivalent.

Return now to the case of mREA. As was already mentioned, this algebra does not possess the coproduct and for constructing the tensor product of mREA modules we cannot use the above scheme as it stands. But it proves to be possible to generalize the final formulae (3.1) and (3.2) to the case of mREA.

Introduce the useful notation for a chain of R-matrices

$$R_i \equiv R_{ii+1}, \qquad R^{\pm 1}_{(i\to j)} \stackrel{\rm def}{=} \begin{cases} R_i^{\pm 1} R_{i+1}^{\pm 1} \ldots R_j^{\pm 1} & \text{if } i < j \\ R_i^{\pm 1} R_{i-1}^{\pm 1} \ldots R_j^{\pm 1} & \text{if } i > j \\ R_i^{\pm 1} & \text{if } i = j. \end{cases} \tag{3.3}$$

The analog of reducible representation (3.1) is established in the following proposition.

[4]Moreover, the algebra $\mathcal{U}$ (as the universal enveloping algebra of any Lie algebra) is a *Hopf* algebra. But for our present purposes only the coproduct is needed.

[5]As is known, a Lie algebra can be always embedded into its universal enveloping algebra. Here x is the image of a Lie algebra generator under such an embedding.

Proposition 2 *Consider the fundamental $\mathcal{L}_q$ module V defined in Proposition 1 and fix a basis e_i, $1 \le i \le n$, in V. The tensor product $V^{\otimes k}$ is also an $\mathcal{L}_q$ module. In the basis $e_{i_1} \otimes \ldots \otimes e_{i_k}$ of $V^{\otimes k}$ the matrices of operators representing the $\mathcal{L}_q$ generators are as follows*

$$\rho_k^t(l_i^j) = \pi^t(l_i^j) \otimes I^{\otimes(k-1)} + \sum_{s=1}^{k-1} R^{-1}_{(s\to 1)} \Big[\pi^t(l_i^j) \otimes I^{\otimes(k-1)}\Big] R^{-1}_{(1\to s)}. \qquad (3.4)$$

Here ρ_k^t and π^t stand for the transposed matrices.

Proof The proposition is proved by direct calculations. Since the calculations are rather lengthy we shall not reproduce them in full detail, giving instead the list of important intermediate steps with the corresponding results.

We prove the proposition by induction in k. For $k = 1$ our assertion reduces to Proposition 1 and hence is true. Suppose it be true up to some $k - 1$ and prove that then it be true for k.

One should verify that operators (3.4) do satisfy the commutation relations (2.2). Let us assign the number $k + 1$ to the auxiliary space of indices of the $\mathcal{L}_q$ generators. One has to show

$$R_{k+1}\rho_k(L_{k+1})R_{k+1}\rho_k(L_{k+1}) - \rho_k(L_{k+1})R_{k+1}\rho_k(L_{k+1})R_{k+1} = \\ R_{k+1}\rho_k(L_{k+1}) - \rho_k(L_{k+1})R_{k+1}. \qquad (3.5)$$

It is convenient to make the transposition of the matrices ρ in the above relation and to take into account that in accordance with (2.3) the matrix $\pi(l_i^j)$ reads

$$\pi^t(L_{k+1})_1 = P_{1\,k+1}B_{k+1},$$

where the unity enumerates the the matrix indices of the representation space. Then, on substituting (3.4) into (3.5) we find that the first summand in the left hand side decomposes into the sum of k^2 terms, a typical one being as follows

$$R_{k+1}R^{-1}_{(n\to 1)}P_{1\,k+1}B_{k+1}\Big[\mathrm{Tr}_{(1)}R^{-1}_{(1\to n)}R^{-1}_{(m\to 1)}B_1R_{1\,k+2}\Big]R^{-1}_{(1\to m)} \equiv R_{k+1}Q(n,m),$$

where the last equality is the definition of $Q(n, m)$, $0 \le n, m \le k - 1$. Here in $Q(0, m)$ the unity matrices are substituted for the chains $R^{-1}_{(1\to n)}$ and $R^{-1}_{(n\to 1)}$. The second summand in the left hand side of (3.5) expands in a similar way but R_{k+1} stands on the right of Q.

In virtue of the supposition of the induction we conclude that it is sufficient to consider just $2k - 1$ terms in each summand containing $Q(k - 1, n)$ and $Q(n, k - 1)$. So, one needs to examine the following expression in the left hand side of (3.5)

$$\sum_{n=0}^{k-2}\Big(R_{k+1}(Q(n,k-1) + Q(k-1,n)) \;-\; (Q(n,k-1) + Q(k-1,n))R_{k+1}\Big) \\ +\; R_{k+1}Q(k-1,k-1) - Q(k-1,k-1)R_{k+1}. \quad (3.6)$$

The proposition will be proved if one could show that this is equal to

$$R_{k+1}R^{-1}_{(k-1\to 1)}P_{1\,k+1}B_{k+1}R^{-1}_{(1\to k-1)} - R^{-1}_{(k-1\to 1)}P_{1\,k+1}B_{k+1}R^{-1}_{(1\to k-1)}R_{k+1}. \qquad (3.7)$$

Consider first the difference, containing $Q(n, k - 1)$, $0 \le n \le k - 2$. Since (see Appendix)

$$\mathrm{Tr}_{(1)}R^{-1}_{(1\to n)}R^{-1}_{(k-1\to 1)}B_1R_{1\,k+2} = R^{-1}_{(k-1\to 2)}P_{2\,k+2}B_{k+2}R^{-1}_{(2\to n+1)} \quad \forall\, n \le k-2$$

and (see, e.g., [17])

$$R_{12}B_1B_2 = B_1B_2R_{12} \tag{3.8}$$

we find

$$\begin{aligned} &R_{k+1}(Q(n,k-1)+Q(k-1,n)) - (Q(n,k-1)+Q(k-1,n))R_{k+1} = \\ &\quad \lambda R^{-1}_{(n\to 1)}R^{-1}_{(k-1\to 1)}P_{1\,k+1}\,B_{k+1}P_{1\,k+2}B_{k+2}R^{-1}_{(2\to k-1)}R^{-1}_{(1\to n)} \\ &\qquad\qquad - \lambda R^{-1}_{(n\to 1)}R^{-1}_{(k-1\to 2)}P_{1\,k+1}B_{k+1}P_{1\,k+2}B_{k+2}R^{-1}_{(1\to k-1)}R^{-1}_{(1\to n)} \end{aligned} \tag{3.9}$$

The calculation of the difference with $Q(k-1,k-1)$ in (3.6) is more involved. The trace contained in $Q(k-1,k-1)$ is as follows (see Appendix)

$$\mathrm{Tr}_{(1)}R^{-1}_{(1\to k-1)}R^{-1}_{(k-1\to 1)}B_1R_{1\,k+2} = I - \lambda P_{2\,k+2}B_{k+2} - \lambda\sum_{n=2}^{k-1}R^{-1}_{(n\to 2)}P_{2\,k+2}B_{k+2}R^{-1}_{(2\to n)}.$$

At last, it is a matter of straightforward calculation to show that the unit matrix I in the above expression leads to the necessary contribution (3.7) while the other terms exactly cancel the unwanted summands of type (3.9). ■

3.2 Decomposition of $V^{\otimes k}$ into mREA submodules

Our next goal is to find the decomposition of the $\mathcal{L}_q$ module $V^{\otimes k}$ into irreducible submodules that is to find an analog of the classical formula (3.2). This can be done on the base of isomorphism $H_k(q) \cong \mathbb{C}[\mathcal{S}_k]$ which was discussed in the first section. For our construction the most important consequence of such an isomorphism is the existence of q-analogs of primitive Young idempotents $\mathcal{Y}_\nu(\sigma) \in H_k(q)$ and decomposition (1.8) of $V^{\otimes k}$ into the direct sum of $H_k(q)$ modules.

The main result of this section is formulated in the following proposition.

Proposition 3 *Consider the mREA $\mathcal{L}_q$ generated by (2.2) with the Hecke R-matrix possessing the symmetry rank p (see (1.3), (1.6) and (1.9)). Let V be the fundamental $\mathcal{L}_q$ module of B type with a fixed basis e_i, $1 \le i \le n$. According to Proposition 2 the space $V^{\otimes k}$ is also an $\mathcal{L}_q$ module for any positive integer k. Decompose the tensor product $V^{\otimes k}$ into the direct sum (1.8).*

Then each component $V_{\nu(a)}$ of the direct sum is an $\mathcal{L}_q$ module and the generators l_i^j are represented by linear operators $\hat{\pi}_{\nu(a)}(l_i^j) \in \mathrm{End}(V_{\nu(a)}) \hookrightarrow \mathrm{End}(V^{\otimes k})$. The matrices of these operators in the basis $e_{i_1}\otimes\ldots\otimes e_{i_k}$ of $V^{\otimes k}$ are of the form

$$\pi^t_{\nu(a)}(l_i^j) = Y_{\nu(a)}(R)\,\rho^t_k(l_i^j)\,Y_{\nu(a)}(R), \tag{3.10}$$

where ρ_k is defined in (3.4) of Proposition 2 and the symbol t means the matrix transposition.

The modules parametrized by different tableaux of the same partition $\nu \vdash k$ are equivalent.

Proof Consider matrices (3.4) of the $\mathcal{L}_q$ representation in $V^{\otimes k}$. Let us assign the number $k+1$ to the auxiliary space of indices of $\mathcal{L}_q$ generators and rewrite the commutation relations in form (3.5). The projectors $Y_{\nu(a)}(R)$ in (3.10) are some polynomials in matrices R_i, $1 \le i \le k-1$. The proof of the proposition is based on the fact that the matrix $\rho^t_k(L_{k+1})$ commute with R_i for $1 \le i \le k-1$. Indeed, taking this for granted for a moment, we conclude that $\rho^t_k(L_{k+1})$ commutes

with all projectors $Y_{\nu(a)}(R)$. Then, due to the orthogonality and the completeness of the set of projectors

$$Y_{\nu(a)}(R)\,Y_{\mu(b)}(R) = \delta_{\nu\mu}\delta_{ab}Y_{\mu(b)}(R), \qquad (3.11)$$
$$\sum_{\nu\vdash k}\sum_{a=1}^{\dim[\nu]} Y_{\nu(a)}(R) = \mathrm{id}_{V^{\otimes k}}$$

it is easy to see that relation (3.5) allows the projection onto each component $V_{\nu(a)}$ of the direct sum (1.8) and the matrices of corresponding representation are given by (3.10).

To prove the commutativity supposed above one needs no particular property of $\pi(l_i^j)$ (an arbitrary $n\times n$ matrix X can be substituted for $\pi(l_i^j)$). The proof consists in a simple calculation and completely based on the Yang-Baxter equation (1.2) and Hecke condition (1.3).

The equivalence of modules parametrized by different Young tableaux of the same partition ν stems from the connection of the corresponding Young idempotents. It can be shown [6] that any two idempotents $\mathcal{Y}_{\nu(a)}(\sigma)$ and $\mathcal{Y}_{\nu(b)}(\sigma)$ are connected by a similarity transform. Namely, there exists an *invertible* element $\mathcal{F}_{ab}(\nu|\sigma)\in H_k(q)$ which is a polynomial in σ_i such that

$$\mathcal{F}_{ab}(\nu|\sigma)\,\mathcal{Y}_{\nu(a)}(\sigma)\,\mathcal{F}_{ab}^{-1}(\nu|\sigma) = \mathcal{Y}_{\nu(b)}(\sigma), \qquad 1\le a,b\le \dim[\nu].$$

In the local representation (1.7) of $H_k(q)$ in $V^{\otimes k}$ the element $\mathcal{F}_{ab}(\nu|\sigma)$ turns into the intertwining operator $F_{ab}(R)$ which ensures the equivalence of $\pi_{\nu(a)}$ and $\pi_{\nu(b)}$

$$F_{ab}\,\pi^t_{\nu(a)}\,F_{ab}^{-1} = F_{ab}\,Y_{\nu(a)}\,\rho^t_k\,Y_{\nu(a)}\,F_{ab}^{-1} = Y_{\nu(b)}\,\rho^t_k\,Y_{\nu(b)} = \pi^t_{\nu(b)}.$$

The second equality in the above chain of transformations is valid due to commutativity of ρ^t_k and $F_{ab}(R)$ since the latter is a polynomial in R_i, $1\le i\le k-1$. ∎

A natural question arises about the irreducibility of modules $V_{\nu(a)}$. Due to some reasons discussed at the end of the paper we suppose the modules to be irreducible, but still we have no general proof of this fact. We state it as a quite plausible hypothesis.

For the representations parametrized by single-row and single-column diagrams the formulae become much simpler and in this case one can explicitly calculate the spectrum of central elements (1.15).

Corollary 3.1 *Consider the $\mathcal{L}_q$modules $V_{\nu(a)}$, defined in Proposition 3.*

i) *For partitions $\nu=(k)$ and $\nu=(1^k)$ (for $k\le p$) the matrices of operators representing the $\mathcal{L}_q$ generators are given by*

$$\pi^t_{(k)}(l_i^j) = q^{1-k}k_q\,S^{(k)}(R)\left[\pi^t(l_i^j)\otimes I^{\otimes(k-1)}\right]S^{(k)}(R), \qquad (3.12)$$

$$\pi^t_{[k]}(l_i^j) = q^{k-1}k_q\,A^{(k)}(R)\left[\pi^t(l_i^j)\otimes I^{\otimes(k-1)}\right]A^{(k)}(R), \quad k\le p, \qquad (3.13)$$

where $S^{(k)}$ and $A^{(k)}$ are the q-symmetrizer and the q-antisymmetrizer correspondingly.

ii) *In the representations $\pi_{(k)}$ and $\pi_{[k]}$ the spectrum χ of central elements $s_m = Tr_qL^m$ takes the following values*

$$\chi_{(k)}(s_m) = q^{-m(p+k-1)-p}\,k_q(p+k-1)_q^{m-1}, \qquad (3.14)$$

$$\chi_{[k]}(s_m) = q^{-m(p-k+1)-p}\,k_q(p-k+1)_q^{m-1}. \qquad (3.15)$$

Proof The assertion **i)** is the direct consequence of (3.10). Indeed, taking into account the relations

$$\begin{aligned} R_i^{\pm 1} S^{(k)}_{12\dots k} &= q^{\pm 1} S^{(k)}_{12\dots k} = S^{(k)}_{12\dots k} R_i^{\pm 1} \\ R_i^{\pm 1} A^{(k)}_{12\dots k} &= -q^{\mp 1} A^{(k)}_{12\dots k} = A^{(k)}_{12\dots k} R_i^{\pm 1} \end{aligned} \qquad 1 \le i \le k-1$$

and the definition (1.5) of a q-number one immediately gets (3.12) and (3.13) from (3.10) where an arbitrary projector $Y_{\nu(a)}$ should be replaced for $S^{(k)}(R)$ or $A^{(k)}(R)$ respectively.

In order to find the values of characters (3.14) and (3.15) we consider $\pi^t_{(k)}(Tr_q L^m)$ (take the q-symmetrical case for the definiteness) and show that this is a multiple of the q-symmetrizer (the identity operator on the subspace $V_{(k)}$). The factor is equal to the character $\chi_{(k)}(s_m)$.

Taking into account (1.11) one rewrites the matrix involved in the form

$$\pi^t_{(k)}(Tr_q L^m) = q^{m(1-k)-2p}\, k_q^m\, S^{(k)}_{12\dots k} \Big[Tr_{(1)} B_1 S^{(k)}_{12\dots k} \Big]^{m-1} S^{(k)}_{12\dots k}.$$

To calculate the trace $Tr_{(1)} B_1 S^{(k)}_{12\dots k}$ we use the recurrent relations for the q-(anti)symmetrizers (see, e.g., [7])

$$S^{(1)}(R) = I \qquad S^{(k)}_{12\dots k}(R) = \frac{1}{k_q}\, S^{(k-1)}_{2\dots k}(R) \Big(q^{1-k} I + (k-1)_q R_{12} \Big) S^{(k-1)}_{2\dots k}(R), \tag{3.16}$$

$$A^{(1)}(R) = I \qquad A^{(k)}_{12\dots k}(R) = \frac{1}{k_q}\, A^{(k-1)}_{2\dots k}(R) \Big(q^{k-1} I - (k-1)_q R_{12} \Big) A^{(k-1)}_{2\dots k}(R) \tag{3.17}$$

where $S^{(k-1)}_{2\dots k}$ is the q-symmetrizer of the $(k-1)$-th order acting in components of $V^{\otimes k}$ with numbers from 2 till k. Then using (2.4) and (1.12) one finds

$$Tr_{(1)} B_1 S^{(k)}_{12\dots k} = q^{-p}\, \frac{(p+k-1)_q}{k_q}\, S^{(k-1)}_{2\dots k}.$$

Having found the above trace one gets

$$\pi^t_{(k)}(Tr_q L^m) = q^{-m(p+k-1)-p}\, k_q (p+k-1)_q^{m-1}\, S^{(k)}(R) = \chi_{(k)}(s_m)\, \mathrm{id}_{V_{(k)}}, \tag{3.18}$$

which proves (3.14). ■

It is worth pointing out that in the above formulae the central role belongs to the symmetry rank p of the R-matrix but not to the parameter[6] n defining the size of R-matrix and the number of generators in the corresponding REA $\mathcal{L}_q$. This feature is characteristic of the Shur-Weyl category connected with the considered series of representations π_ν [14].

3.3 The sl-reduction

As the next aim, we are going to explore a problem of tensor product of fundamental $\mathcal{SL}_q$ representations (2.10). Actually, the solution for the problem follows from Proposition 3 and (2.8). The only thing we need is the spectrum of the element $Tr_q L$ in representations (3.10).

Lemma 1 *Let the partition $\nu \vdash k$ be of the height s that is*

$$\nu = (\nu_1, \nu_2, \dots, \nu_s), \quad \sum_{r=1}^{s} \nu_i = k, \quad \nu_1 \ge \nu_2 \ge \dots \ge \nu_s > 0.$$

[6] As was already mentioned above there exist examples of R-matrices with $p \ne n$ [7].

Then the spectrum of the central element $s_1 = Tr_q L$ in the representation $\pi_{\nu(a)}$ $1 \le a \le \dim[\nu]$ is as follows

$$\chi_\nu(s_1) = q^{-2p} \sum_{r=1}^{s} q^{2r-1-\nu_r} (\nu_r)_q, \tag{3.19}$$

where p is the symmetry rank of R-matrix and $(\nu_r)_q$ is the q-analog of the integer ν_r (see definition (1.5)).

Proof Let us find the matrix $\pi_{\nu(a)}(Tr_q L)$ for an arbitrary $\nu(a)$. Using (1.14) and (1.11) one immediately gets from (3.10)

$$\pi^t_{\nu(a)}(Tr_q L) = q^{-2p}\, Y_{\nu(a)}(R)\Big(J_1^{-1} + J_2^{-1} + \ldots + J_k^{-1}\Big) Y_{\nu(a)}(R).$$

Here the matrices J_k read

$$J_1 = I, \quad J_i = R_{(i-1\to 1)} R_{(1\to i-1)} \quad i \ge 2. \tag{3.20}$$

They are the images of the *Jucys-Murphy* elements $\mathcal{J}_i(\sigma)$ of the Hecke algebra $H_k(q)$ under the local representation (1.7). The elements $\mathcal{J}_i(\sigma)$ generate the maximal commutative subalgebra in $H_k(q)$ and have the following important property (see, e.g., [6])

$$\mathcal{J}_i(\sigma)\mathcal{Y}_{\nu(a)}(\sigma) = \mathcal{Y}_{\nu(a)}(\sigma)\mathcal{J}_i(\sigma) = q^{2(c_i - r_i)}\mathcal{Y}_{\nu(a)}(\sigma),$$

where c_i and r_i are respectively the coordinates of the column and the row to which the box with the number i belongs. The quantity $q^{2(c-r)}$ is called a *content* of the (c, r)-th box of a given Young diagram. Here is an example of the diagram $\nu = (4, 3, 1^2)$ with corresponding contents

1	q^2	q^4	q^6
q^{-2}	1	q^2	
q^{-4}			
q^{-6}			

Therefore we come to the result

$$\pi^t_{\nu(a)}(Tr_q L) = q^{-2p}\Big(\sum_{i=1}^{k} q^{-2(c_i - r_i)}\Big) Y_{\nu(a)} = \chi_\nu(Tr_q L) Y_{\nu(a)}. \tag{3.21}$$

It is obvious that the sum of all contents (or their inverses) of a given tableau $\nu(a)$ is completely defined by the partition (diagram) ν and has the same value for all tableaux $\nu(a)$ corresponding to a given ν. As a consequence, the character $\chi_\nu(Tr_q L)$ does not depend on a. Of course, this is in agreement with the equivalence of representations $\pi_{\nu(a)}$ parametrized by different tableaux $\nu(a)$ of the same partition ν.

It is a matter of a simple calculation to show that $\chi_\nu(Tr_q L)$ represented as the sum of all inverse contents of the tableau $\nu_{(a)}$ can be written in form (3.19). ■

It is interesting to point out one peculiarity of the quantum case. Namely, the spectrum of central elements distinguishes the representations more effectively than in the classical case. To show this, take the q as a parameter and extend the complex field $\mathbb{C}$ to the field of rational functions in q. The character $\chi_\nu(s_1)$ can be identically transformed to the following expression

$$q^{2p}\lambda\, \chi_\nu(s_1) = q^s s_q - \sum_{r=1}^{s} q^{2(r-\nu_r)-1} \equiv q^s s_q - \Omega(\nu_1, \ldots, \nu_s).$$

Given $\Omega(\nu_1, \ldots, \nu_s)$ as a function in q, one unambiguously restores the values of all ν_r, since the numbers $2(r - \nu_r)$ form a *strictly* increasing sequence. Therefore if the height s of a partition ν is fixed and known then the spectrum of the first central element $s_1 = Tr_q L$ is sufficient to distinguish the representations. It is not so, of course, for the classical case $q \to 1$.

Now we are in a position to formulate the results for the higher dimensional modules of the algebra $\mathcal{SL}_q$.

Proposition 4 *Consider the algebra $\mathcal{SL}_q$ (2.7) with a Hecke R-matrix possessing the symmetry rank p. Let V be the fundamental $\mathcal{L}_q$ module of B type with a fixed basis e_i, $1 \le i \le n$. According to Proposition 2 the tensor product $V^{\otimes k}$ is also an $\mathcal{L}_q$ module for any $k \in \mathbb{N}$. The following assertions are true.*

i) *The space $V^{\otimes k}$ is a reducible $\mathcal{SL}_q$ module. In the basis $e_{i_1} \otimes \ldots \otimes e_{i_k}$ the matrices of operators representing the $\mathcal{SL}_q$ generators f_i^j have the following form*

$$\bar{\rho}_k^t(f_i^j) = \frac{1}{\omega}\Big(\rho_k^t(l_i^j) - \frac{\delta_i^j}{p_q q^p}\mathcal{Z}_k\Big), \qquad \omega = \frac{q^{1-p}}{p_q}\,(q^{p-2}(p+1)_q - 1), \tag{3.22}$$

where the symbol t means the matrix transposition, $\rho_k(l_i^j)$ is defined in (3.4) and $\mathcal{Z}_k$ is given by

$$\mathcal{Z}_k = I + \sum_{n=1}^{k-1} R^{-1}_{(n\to 1)} R^{-1}_{(1\to n)}.$$

ii) *Decompose the tensor product $V^{\otimes k}$ into the direct sum (1.8). Each component $V_{\nu(a)}$ of the direct sum is an $\mathcal{SL}_q$ submodule in $V^{\otimes k}$. The generators f_i^j are represented by linear operators with the following matrices*

$$\bar{\pi}^t_{\nu(a)}(f_i^j) = \frac{1}{\omega_\nu}\, Y_{\nu(a)}\Big[\rho_k^t(l_i^j) - \delta_i^j\,\frac{q^p}{p_q}\,\chi_\nu(s_1) I_{12\ldots k}\Big] Y_{\nu(a)}, \qquad \omega_\nu = 1 - \lambda\,\frac{q^p}{p_q}\,\chi_\nu(s_1), \tag{3.23}$$

the character $\chi_\nu(s_1)$ being defined in (3.19). The modules parametrized by different tableaux $\nu(a)$ of the same partition $\nu \vdash k$ are equivalent.

iii) *The spectrum $\bar{\chi}$ of the $\mathcal{SL}_q$ central elements $\bar{s}_m = Tr_q F^m$ in representations $\bar{\pi}_{(k)}$ and $\bar{\pi}_{[k]}$ corresponding to $\nu = (k)$ and $\nu = (1^k)$ takes the following values*

$$\bar{\chi}_{(k)}(\bar{s}_m) = q^{-p-m}\,\frac{k_q(p-1)_q(p+k)_q}{(p+k-1)_q}\,\frac{\Big[(p-1)_q^{m-1}(p+k)_q^{m-1} + (-1)^m k_q^{m-1}\Big]}{(q^{p-2}(p+k)_q - k_q)^m} \tag{3.24}$$

$$\bar{\chi}_{[k]}(\bar{s}_m) = q^{-p+m}\,\frac{k_q(p+1)_q(p-k)_q}{(p-k+1)_q}\,\frac{\Big[(p+1)_q^{m-1}(p-k)_q^{m-1} + (-1)^m k_q^{m-1}\Big]}{(q^{p+2}(p-k)_q + k_q)^m} \tag{3.25}$$

Proof This proposition is the direct consequence of Propositions 2 and 3, Corollary 3.1 and rule (2.8). Indeed, the operators $\bar{\rho}_k(Tr_q F)$ and $\bar{\pi}_{\nu(a)}(Tr_q F)$ are obviously equal to zero. Basing on Propositions 2 and 3 one can verify that (3.22) and (3.23) do satisfy (2.2) and that the factor ω_ν^{-1} ensures the proper normalization of the right hand side of (2.2).

As for the values (3.24) and (3.25), they can be found by straightforward but rather lengthy calculations on the base of (3.23), (3.14) and (3.15). ■

In the case of $U_q(sl_n)$ R-matrix we have $p = n$ and at the classical limit $q \to 1$ the spectrum (3.24), (3.25) of the $\mathcal{SL}_q$ central elements tends to the spectrum of the $U(sl_n)$ Casimir elements in the corresponding representations (see, e.g., [18]).

At last, consider the tensor product of two (irreducible) modules V_μ and V_ν over $\mathcal{L}_q$ (or $\mathcal{SL}_q$) in which the representation operators π_μ and π_ν are given by (3.10) (or by (3.23)). Using the isomorphism $H_k \cong \mathbb{C}[\mathcal{S}_k]$ one can show (see, e.g., [14]) that the tensor product $V_\mu \otimes V_\nu$ is isomorphic to the following direct sum of $\mathcal{L}_q$ (or $\mathcal{SL}_q$) modules V_σ

$$V_\mu \otimes V_\nu \cong c^\sigma_{\mu\nu} V_\sigma. \tag{3.26}$$

Here $c^\sigma_{\mu\nu}$ are the Littlewood-Richardson coefficients defining a ring structure in the set of Schur symmetric functions.

4 Fundamental module of R type

Besides the fundamental REA module of B type considered in the previous sections, one can construct another module which will be called *the fundamental module of R type*. In the case of R-matrix connected with $U_q(sl_n)$ the corresponding representation originates from the general theory of dual Hopf algebras and we generalize it to the case of an arbitrary R-matrix.

4.1 The definition and tensor product decomposition rule

So, suppose at first that R-matrix defining the structure of REA is the Drinfeld-Jimbo R-matrix connected with the quantum universal enveloping algebra $U_q(sl_n)$. In this case the Hopf algebra (1.16) is an algebra $\mathrm{Fun}_q(GL(n))$ of functions on the quantum group [5]. Besides, there exists embedding (1.19) of the corresponding REA $\mathcal{L}_q$ into $U_q(gl_n)$ — the dual Hopf algebra to $\mathrm{Fun}_q(GL(n))$. As a consequence, it is possible to define a pairing among the generators $\hat{l}_i^{\,j}$ and $t_i^{\,j}$.

Using the explicit formulae for the paring of $U_q(gl_n)$ generators $L^\pm$ and $\mathrm{Fun}_q(GL(n))$ generators T (see [5]), we get the following result

$$\langle T_1 T_2 \dots T_k, \hat{L}_{k+1} \rangle = R_{(k\to 1)} R_{(1\to k)} \equiv J_{k+1}. \tag{4.1}$$

Here we have used the compact notations (3.3) and (3.20) for the chains of R-matrices.

As is known from the Hopf algebra theory, any module over a Hopf algebra $\mathcal{H}$ can be transformed into a comodule over its dual Hopf algebra $\mathcal{H}^*$ and vice versa. Let us use this fact in order to define a representation of REA $\mathcal{L}_q$ in a finite dimensional vector space V, $\dim V = n$.

On fixing a basis e_i, $1 \le i \le n$, one can convert the space V into a *left* comodule over the Hopf algebra $\mathcal{H} = \mathrm{Fun}_q(GL(n))$ by means of the corepresentation δ

$$\delta : \; V \to \mathcal{H} \otimes V, \qquad \delta(e_i) = t_i^{\,j} \otimes e_j$$

where the summation over the repeated indices is understood. Then in V we get a *right* $\mathcal{L}_q$ action by the following rule

$$e_1 \triangleleft \hat{L}_2 = \langle T_1, \hat{L}_2 \rangle \, e_1 = R^2_{12} \, e_1.$$

This action can be easily expanded to the tensor product $V^{\otimes k}$ for any $k \in \mathbb{N}$. Indeed the comodule structure of $V^{\otimes k}$ is obvious

$$\delta_k : \; V^{\otimes k} \to \mathcal{H} \otimes V^{\otimes k}, \qquad e_1 \otimes \ldots \otimes e_k \xrightarrow{\delta_k} T_1 \ldots T_k \otimes (e_1 \otimes \ldots \otimes e_k).$$

Hence, taking into account (4.1)

$$e_1 \otimes \ldots \otimes e_k \triangleleft \hat{L}_{k+1} = J_{k+1}\, e_1 \otimes \ldots \otimes e_k.$$

It turns out that we can directly generalize the above formulae to the case of an *arbitrary* R-matrix. The following proposition is easy to verify.

Proposition 5 *Consider the REA $\mathcal{L}_q$ generated by relations (1.1) with an arbitrary R-matrix. The matrix will be treated as that of a linear operator acting in the tensor square of a finite dimensional vector space V, $\dim V = n$. Define a linear map $\theta_k : \mathcal{L}_q \to \mathrm{End}(V^{\otimes k})$ by the following rule*

$$\left\{\begin{array}{l} \theta_k(e_{\mathcal{L}}) = \mathrm{id}_{V^{\otimes k}} \\ \theta_k(\hat{L}_{k+1}) = \alpha\, J_{k+1} \\ \theta_k(\hat{l}_1 \cdot \hat{l}_2 \cdot \ldots \cdot \hat{l}_m) = \theta_k(\hat{l}_1) \cdot \theta_k(\hat{l}_2) \cdot \ldots \cdot \theta_k(\hat{l}_m), \end{array}\right. \tag{4.2}$$

where $\alpha \neq 0$ is an arbitrary complex number. Then θ_k realizes a representation of $\mathcal{L}_q$ in the space $V^{\otimes k}$.

Proof It is sufficient to substitute the matrices $\theta_k(\hat{L}_{k+1}) = \alpha J_{k+1}$ in (1.1) rewritten in the form

$$R_{k+1}\hat{L}_{k+1}R_{k+1}\hat{L}_{k+1} - \hat{L}_{k+1}R_{k+1}\hat{L}_{k+1}R_{k+1} = 0$$

and make use of the following consequence of Yang-Baxter equation (1.2)

$$\begin{array}{ll} (R_1 \ldots R_k)R_i = R_{i+1}(R_1 \ldots R_k), & 1 \le i \le k-1 \\ (R_k \ldots R_1)R_i = R_{i-1}(R_k \ldots R_1), & 2 \le i \le k. \end{array} \tag{4.3}$$

As for the numeric factor $\alpha \neq 0$, it can be arbitrary due to the renormalization automorphism $\hat{L} \to \alpha \hat{L}$ (see Remark 1). ■

Note that in proving Proposition 5 we use nothing but the Yang-Baxter equation for the R-matrix. Therefore, representation (4.2) is valid not only for the quantum group R-matrix but also for an arbitrary solution of the Yang-Baxter equation (even of a non-Hecke type).

At $k > 1$ the representation θ_k is reducible. The Hecke condition (1.3) is needed for extracting the irreducible components of θ_k.

Proposition 6 *Let REA $\mathcal{L}_q$ be generated by (1.1) with a Hecke R-matrix. Consider the representation θ_k (4.2) in the space $V^{\otimes k}$. Decompose $V^{\otimes k}$ into the direct sum of subspaces $V_{\nu(a)}$ in accordance with (1.8).*

Then each $V_{\nu(a)}$ is an $\mathcal{L}_q$ submodule and the matrices of linear operators representing the generators $\hat{l}_i^j$ are given by

$$\theta_{\nu(a)}(\hat{L}_{k+1}) = Y_{\nu(a)}(R)\, \theta_k(\hat{L}_{k+1})\, Y_{\nu(a)}(R). \tag{4.4}$$

The modules parametrized by different tableaux of the same partition $\nu \vdash k$ are equivalent.

Proof The proof is based on the fact that q-projectors $Y_\nu(R)$ are actually polynomials in $J_1, \ldots, J_k$ for $\nu \vdash k$ (see [6]). Being the images of Jucys-Murphy elements, the operators J_i commute[7] with J_{k+1}. As a consequence, the relation

$$R_{k+1}\theta_k(\hat{L}_{k+1})R_{k+1}\theta_k(\hat{L}_{k+1}) = \theta_k(\hat{L}_{k+1})R_{k+1}\theta_k(\hat{L}_{k+1})R_{k+1}$$

[7] One can verify this fact independently, on the base of (4.3).

which takes place in $\mathrm{End}(V^{\otimes k})$ admits projection into subspaces $\mathrm{End}(V_\nu)$ in accordance with (4.4).

The equivalence of $V_{\nu(a)}$ and $V_{\nu(b)}$ corresponding to different tableaux of the same partition ν is proved in the same way as in Proposition 3. ■

It is worth mentioning, that for constructing representations θ_ν one has no need the symmetry rank of R to be finite.

As in the case of B type module one can explicitly calculate the spectrum of central elements (1.15) in the representations parametrized by single-row and single-column diagrams.

Corollary 6.1 *In the representations parametrized by partitions $\nu = (k)$ and $\nu = (1^k)$ the spectrum $\hat\chi$ of the central elements $s_m = Tr_q \hat L^m$ takes the following values*

$$\hat\chi_{(k)}(s_m) = q^{-p}\Big(q^{-2m}p_q + \lambda\,\frac{(p+k)_q}{(k+1)_q}\,q^{m(k-1)}[m(k+1)]_q\Big) \tag{4.5}$$

$$\hat\chi_{[k]}(s_m) = q^{-p}\Big(q^{2m}p_q - \lambda\,\frac{(p-k)_q}{(k+1)_q}\,q^{-m(k-1)}[m(k+1)]_q\Big) \qquad k\le p \tag{4.6}$$

Proof We shall consider the case of q-symmetric representation $\theta_{(k)}$ for the definiteness. Let us first calculate $\theta_{(k)}(Tr_q\hat L)$. In accordance with (4.2) and (4.4) the matrices representing REA generators read

$$\theta_{(k)}(\hat L_{k+1}) = S^{(k)}J_{k+1}S^{(k)}.$$

Taking into account (see [6]) that

$$S^{(k+1)} = S^{(k)}\,\frac{J_{k+1}-q^{-2}}{q^{2k}-q^{-2}} \tag{4.7}$$

we rewrite the matrix $\theta_{(k)}(\hat L_{k+1})$ in the equivalent form

$$\theta_{(k)}(\hat L_{k+1}) = \lambda q^{k-1}(k+1)_q S^{(k+1)} + q^{-2}S^{(k)}.$$

Next, in virtue of

$$Tr_{q(k+1)}S^{(k+1)} = q^{-p}\,\frac{(p+k)_q}{(k+1)_q}\,S^{(k)}$$

we come to the final result

$$\theta_{(k)}(Tr_q\hat L) = q^{-p}\left(q^{-2}p_q + \lambda q^{k-1}(p+k)_q\right)\,S^{(k)} \equiv \hat\chi_{(k)}(s_1)\,\mathrm{id}_{V_{(k)}}. \tag{4.8}$$

Then, basing on (4.7) and on the relations

$$S^{(k+1)}S^{(k)} = S^{(k+1)}, \qquad S^{(k+1)}J_{k+1} = q^{2k}S^{(k+1)}$$

one can prove (4.5) by induction in the power m of $Tr_q\hat L^m$, where (4.8) serves as the first step. The final step of induction gives

$$\hat\chi_{(k)}(s_m) = q^{-p}\Big(q^{-2m}p_q + \lambda q^{m(k-1)}(p+k)_q\sum_{r=0}^{m-1}q^{(k+1)(2r+1-m)}\Big).$$

With substitution $t = q^{k+1}$ one can easily show that

$$\sum_{r=0}^{m-1}q^{(k+1)(2r+1-m)} = \frac{[m(k+1)]_q}{(k+1)_q}$$

coming thereby to the desired result (4.5). ∎

The representation θ_1 in the space V itself is irreducible and the matrices of operators representing $\mathcal{L}_q$ generators are as follows

$$\theta_1(\hat{L}_2) = R^2_{12}. \tag{4.9}$$

where the indices of the first space are those of matrices from $\mathrm{End}(V)$ and the indices of the second space enumerates the generators of the algebra. The space V with the above $\mathcal{L}_q$ representation will be called *the fundamental module of R type.*

The representation of mREA (2.2) obtained from (4.9) by shift (2.1) reads

$$\theta_1(L_2) = -R_{12}, \tag{4.10}$$

where we first perform a renormalization of (4.9) by the factor $\alpha = -\lambda^{-1}$.

4.2 The sl-reduction

In order to pass from the REA representation $\theta_{\nu(a)}$ (4.4) to the corresponding representation $\bar{\theta}_{\nu(a)}$ of the algebra $\mathcal{SL}_q$ we need to calculate the spectrum of $\theta_{\nu(a)}(Tr_q\hat{L})$.

Lemma 2 *Let the partition $\nu \vdash k$ be of the height s that is*

$$\nu = (\nu_1, \nu_2, \dots, \nu_s), \quad \sum_{r=1}^{s} \nu_i = k, \quad \nu_1 \geq \nu_2 \geq \ldots \geq \nu_s > 0.$$

Then the spectrum of the central element $s_1 = Tr_q\hat{L}$ in the representation $\theta_{\nu(a)}$ $1 \leq a \leq \dim[\nu]$ is as follows

$$\theta_{\nu(a)}(Tr_q\hat{L}) = \zeta_\nu(s_1)Y_{\nu(a)}, \qquad \zeta_\nu(s_1) = q^{-p}p_q + \lambda\sum_{r=1}^{s} q^{\nu_r+1-2r}(\nu_r)_q, \tag{4.11}$$

where p is the symmetry rank of R-matrix and $(\nu_r)_q$ is the q-analog of the integer ν_r (see definition (1.5)).

Proof The lemma is proved by the straightforward calculation in analogy with the proof of Lemma 1. ∎

Let us consider the representations $\theta_{(k)}$ and $\theta_{[k]}$ corresponding to single-row and single-column diagrams in more detail. In this case one can explicitly calculate the spectrum of central elements similarly to the B type representation.

Proposition 7 *Let the Hecke type R-matrix has the symmetry rank p. Consider the REA representations of R type $\theta_{(k)}$ and $\theta_{[k]}$ parametrized by partitions $\nu = (k)$ and $\nu = (1^k)$ (4.4). Then the corresponding representations $\bar{\theta}$ of the $\mathcal{SL}_q$ generators f_i^j are as follows*

$$\bar{\theta}_{(k)}(F_{k+1}) = \frac{q^{1-p}(p+k)_q}{q^{2-p}(p+k)_q - k_q}\Big(S^{(k)}I_{k+1} - \frac{p_q(k+1)_q}{(p+k)_q}\,S^{(k+1)}\Big) \tag{4.12}$$

$$\bar{\theta}_{[k]}(F_{k+1}) = \frac{q^{-1-p}(p-k)_q}{q^{-2-p}(p-k)_q + k_q}\Big(\frac{p_q(k+1)_q}{(p-k)_q}\,A^{(k+1)} - A^{(k)}I_{k+1}\Big) \tag{4.13}$$

The spectrum $\bar\zeta$ of the $\mathcal{SL}_q$ central elements $\bar s_m = Tr_q F^m$ in these representations takes the following values

$$\bar\zeta_{(k)}(\bar s_m) = q^{-p-m(p-1)}\,\frac{k_q(p-1)_q(p+k)_q}{(k+1)_q}\,\frac{\left[(p+k)_q^{m-1} + (-1)^m k_q^{m-1}(p-1)_q^{m-1}\right]}{(q^{2-p}(p+k)_q - k_q)^m} \tag{4.14}$$

$$\bar\zeta_{[k]}(\bar s_m) = q^{-p-m(p+1)}\,\frac{k_q(p+1)_q(p-k)_q}{(k+1)_q}\,\frac{\left[(-1)^m(p-k)_q^{m-1} + k_q^{m-1}(p+1)_q^{m-1}\right]}{(q^{-2-p}(p-k)_q + k_q)^m} \tag{4.15}$$

Proof The proof consists in direct calculations on the base of (2.8) and we shall not present it here. ■

4.3 Interrelation between modules of B and R types

Let us now find a connection between the fundamental modules of B and R types. If the symmetry rank $p = 2$ (for example, when R stems from $U_q(sl_2)$) these modules are equivalent. To be more precise, the situation is as follows. In virtue of (1.9) the q-antisymmetrizer $A^{(2)}$ is a unit rank projector in $V^{\otimes 2}$ and its matrix can be written in the form

$$A_{i_1i_2}^{j_1j_2} = u_{i_1i_2}v^{j_1j_2},$$

the matrices $\|u_{ij}\|$ and $\|v^{ij}\|$ being nonsingular. Then one can show that the representations π and θ_1 of mREA (2.2) are connected by the relation

$$q^2\, u_1\cdot\pi(L_2)\cdot u_1^{-1} = qI_{12} + \theta_1(L_2).$$

After the sl-reduction we come to the representations $\bar\pi$ and $\bar\theta_1$ of the algebra $\mathcal{SL}_q$ (2.7) and simplify the above formula to the expression

$$u_1\cdot\bar\pi(F_2)\cdot u_1^{-1} = \bar\theta_1(F_2),$$

$F = \|f_i^{\,j}\|$ being the matrix composed of the $\mathcal{SL}_q$ generators.

In the case $p > 2$ the fundamental modules of B and R types are not equivalent. Constraining ourselves to the case of $\mathcal{SL}_q$ algebra (2.7) we shall prove that R type representation $\bar\theta(F)$ is equivalent to $\bar\pi_{[p-1]}(F)$ obtained from (3.13) by means of sl-reduction (3.23).

For this purpose, consider in more detail the structure of the subspace $V_{[p-1]}\subset V^{\otimes(p-1)}$. By definition (1.8) the subspace $V_{[p-1]}$ is the image of the q-antisymmetrizer $A^{(p-1)}$

$$V_{[p-1]} = A^{(p-1)}(R)\triangleright V^{\otimes(p-1)}.$$

Since the symmetry rank of R-matrix is equal to p then the q-antisymmetrizer $A^{(p)}$ is a unit rank projector and its matrix can be written in the form

$$A^{(p)}{}_{i_1\dots i_p}^{\,j_1\dots j_p} = u_{i_1\dots i_p}v^{j_1\dots j_p} \tag{4.16}$$

where as follows from (3.11) the tensors u and v are normalized by the condition

$$\sum_{\{i\}} u_{i_1\dots i_p}v^{i_1\dots i_p} = 1.$$

It is convenient to introduce the following linear combinations of the basis vectors of the space $V^{\otimes(p-1)}$

$$\epsilon^i \stackrel{\rm def}{=} \sum_{\{a\}} v^{ia_2\dots a_p}e_{a_2}\otimes\ldots\otimes e_{a_p}. \tag{4.17}$$

The following lemma establishes an important property of the vectors ϵ^i.

Lemma 3 *Consider the set of n vectors $\epsilon^i \in V^{\otimes(p-1)}$ defined in (4.17). These are eigenvectors of the q-antisymmetrizer $A^{(p-1)}$ and they form a basis of the subspace $V_{[p-1]}$*

$$A^{(p-1)}(R) \triangleright \epsilon^i = \epsilon^i, \qquad \forall\, \mathbf{w} \in V_{[p-1]}: \quad \mathbf{w} = \sum_i w_i \epsilon^i. \tag{4.18}$$

Proof Consider the recurrence relation (3.17) for the q-antisymmetrizer $A^{(p)}$ and calculate the trace in the first matrix space with the matrix B_1

$$Tr_{(1)} B_1 A^{(p)}_{12\dots p} = \frac{1}{q^p p_q}\, A^{(p-1)}_{2\dots p}.$$

In virtue of (4.16) we get the following expression for the matrix $\mathbb{A}$ of $A^{(p-1)}$

$$\mathbb{A}_{a_2\dots a_p}^{\;b_2\dots b_p} = q^p p_q \sum_{m,n} B^{\,n}_{m} u_{na_2\dots a_p} v^{mb_2\dots b_p}, \tag{4.19}$$

where for the compactness we omit the superscript $(p-1)$ of the matrix $\mathbb{A}$. Also we need the formula connecting the matrix C (1.10) and the tensors u and v. It can be shown (see [7]) that

$$C_i^{\,j} = \frac{p_q}{q^p} \sum_{\{a\}} u_{ia_2\dots a_p} v^{ja_2\dots a_p} \equiv \frac{p_q}{q^p} \sum_{\{a\}} u_{i\{a\}} v^{j\{a\}}, \tag{4.20}$$

where in the last equality we have introduced a convenient multi-index notation.

At last, taking into account definition (4.17) we get the necessary result (the summation over the repeated indices is understood)

$$\begin{aligned} A^{(p-1)} \triangleright \epsilon^i &= v^{ia_2\dots a_p} \mathbb{A}_{a_2\dots a_p}^{\;b_2\dots b_p} e_{b_2} \otimes \dots \otimes e_{b_p} \equiv v^{i\{a\}} \mathbb{A}_{\{a\}}^{\{b\}} \mathbf{e}_{\{b\}} \\ &= q^p p_q\, v^{i\{a\}}\, \mathbf{e}_{\{b\}}\, v^{m\{b\}}\, B^{\,n}_{m}\, u_{n\{a\}} = q^p p_q\, \epsilon^m B^{\,n}_{m} u_{n\{a\}} v^{i\{a\}} \\ &= q^{2p}\, \epsilon^m\, B^{\,n}_{m} C^{\,i}_{n} = \epsilon^i. \end{aligned}$$

Here at the last step we have used (1.11).

Therefore, under the action of the q-antisymmetrizer $A^{(p-1)}$ the space $W = \mathrm{Span}\{\epsilon^i\}$ is an invariant subspace in $V_{[p-1]}$ and hence $W = V_{[p-1]}$. But as was proved in [7]

$$\dim V_{[p-1]} = \dim V = n.$$

Therefore the n vectors ϵ^i cannot be linear dependent since otherwise $\dim V_{[p-1]} = \dim W < n$. So, the set of eigenvector ϵ^i of the q-antisymmetrizer $A^{(p-1)}$ can be taken a basis of $V_{[p-1]}$. ■

Now we are ready to establish the connection of B and R type fundamental modules in the case of R-matrix with a finite symmetry rank.

Proposition 8 *Let the R-matrix has the symmetry rank p. Then the $\mathcal{SL}_q$ representation $\bar\theta_1$ obtained from (4.10) is equivalent to $\bar\pi_{[p-1]}$ obtained from (3.13) by sl-reduction (3.23).*

Proof Let us first consider the $\mathcal{L}_q$ representation $\pi_{[p-1]}$ (3.13) which acts in the subspace $V_{[p-1]}$. In virtue of Lemma 3 we shall find the matrices of operators $\pi_{[p-1]}(l_i^{\,j})$ in the basis of vectors ϵ^k (4.17). Using (3.13) we obtain (in the same notations as in the proof of Lemma 3)

$$\begin{aligned} \frac{q^{2-p}}{(p-1)_q} \pi_{[p-1]}(l_i^{\,j}) \triangleright \epsilon^k &= v^{k\{a\}} \mathbb{A}_{\{a\}}^{m\{c\}} B^{\,j}_{m} \mathbb{A}_{i\{c\}}^{\{b\}} \mathbf{e}_{\{b\}} = (\text{use } (4.19)) \\ &= q^{2p} p_q^2\, \mathbf{e}_{\{b\}} v^{r\{b\}} B^{\,l}_{s} (v^{k\{a\}} u_{l\{a\}}) B^{\,n}_{r} (v^{sm\{c\}} u_{ni\{c\}}) B^{\,j}_{m} \\ &= q^{3p} p_q\, \epsilon^r\, (B^{\,l}_{s} C^{\,k}_{l}) B^{\,n}_{r} (v^{sm\{c\}} u_{ni\{c\}}) B^{\,j}_{m} = q^p p_q\, \epsilon^r B^{\,n}_{r} (v^{km\{c\}} u_{ni\{c\}}) B^{\,j}_{m}. \end{aligned}$$

Introduce an $n^2 \times n^2$ matrix Ω with matrix elements

$$\Omega_{s_1 s_2}^{\,r_1 r_2} = p_q (p-1)_q \sum_{\{a\}} u_{s_1 s_2 \{a\}} v^{r_1 r_2 \{a\}}.$$

Then, the matrix of the operator $\pi_{[p-1]}(l_i^j)$ in the basis ϵ^k has the form (in compact notations)

$$(\pi_{[p-1]}(L_2))_1 = q^{2(p-1)}\, B_1 \Omega_{12} B_2. \tag{4.21}$$

With the use of (3.17) for $A^{(p)}$ and (4.20) for C one can express the matrix Ω in a more explicit form. Omitting straightforward calculations we write down the final result

$$\Omega_{12} = q^{2p-1}\Big(C_1 C_2 - q C_1 \Psi_{21} C_1\Big),$$

where Ψ is the skew-inverse to R-matrix as defined in (1.6). Substituting this into (4.21) we find

$$(\pi_{[p-1]}(L_2))_1 = q^{-3} I_{12} - q^{2p-2} \Psi_{21} C_1 B_2.$$

After sl-reduction (3.23) we get the representation of the $\mathcal{SL}_q$ algebra

$$(\bar\pi_{[p-1]}(F_2))_1 = \frac{q^{1-p}}{(p-1)_q + q^{p+2}}\,(I_{12} - q^{3p} p_q \Psi_{21} C_1 B_2).$$

The sl-reduction of the R type representation (4.10) leads in turn to the result

$$\bar\theta_1(F_2) = \frac{q^{p+1}}{(p-1)_q + q^{p+2}}\,(I_{12} - q^{-p} p_q\, R_{12}).$$

Next we take into account the connection of Ψ and R (see Appendix)

$$q^{2p} C_1 \Psi_{21} B_2 = R_{12}^{-1}. \tag{4.22}$$

With this formula one immediately gets

$$C_1\, (\bar\pi_{[p-1]}(F_2))_1\, C_1^{-1} = \bar\theta_1(F_2)$$

which means that the corresponding modules are equivalent. ■

4.4 Indecomposable modules: an example

One can put a natural question about the completeness of the set of representations thus obtained. In other words, whether an arbitrary finite dimensional REA module with a non-commutative representation of (1.1) is equivalent to a direct sum of modules V_ν defined in Proposition 3?

The answer to this question is negative. The matter is that REA (1.1) possesses reducible finite dimensional but *indecomposable* modules. The corresponding mREA representations do not admit a finite classical limit $q \to 1$. Let us give a simplest example of such a module for algebra $\mathcal{L}_q$ (1.13).

We start from the one-dimensional representation $\rho :\, \mathcal{L}_q \to \mathbb{C}$ (see [8])

$$\rho(\hat L) = \begin{pmatrix} 0 & x \\ y & z \end{pmatrix}, \qquad x, y, z \in \mathbb{C}. \tag{4.23}$$

Then we use the comodule property (1.18) in order to get the higher dimensional representation of $\mathcal{L}_q$. For this purpose take the known R-matrix representation γ of (1.16)

$$\gamma(T_1) = P_{12} R_{12}, \quad \gamma(S(T_1)) = R_{12}^{-1} P_{12}.$$

Here P is the transposition matrix, R is the $U_q(sl_2)$ R-matrix and the second matrix space stands for the representation space V, $\dim V = 2$. Then in accordance with (1.18) we construct a two dimensional representation ρ_2 of $\mathcal{L}_q$ in the space $V \otimes \mathbb{C} \cong V$

$$\rho_2(\hat{L}_1) = \gamma(T_1)\rho(\hat{L}_1)\gamma(S(T_1)) = R_{21}\rho(\hat{L}_2)R_{21}^{-1}.$$

For the $\mathcal{L}_q$ generators (1.13) the explicit form of the representation ρ_2 reads as follows

$$\rho_2(\hat{a}) = \begin{pmatrix} 0 & -q\lambda x \\ 0 & 0 \end{pmatrix}, \ \rho_2(\hat{b}) = \begin{pmatrix} qx & 0 \\ 0 & q^{-1}x \end{pmatrix}, \ \rho_2(\hat{c}) = \begin{pmatrix} q^{-1}y & -\lambda z \\ 0 & qy \end{pmatrix}, \ \rho_2(\hat{d}) = \begin{pmatrix} z & q^{-1}\lambda x \\ 0 & z \end{pmatrix}.$$

The module V with the representation ρ_2 is reducible. The one-dimensional submodule is spanned by the basis vector e_1. The corresponding one-dimensional representation

$$\hat{a} \to 0, \quad \hat{b} \to qx, \quad \hat{c} \to q^{-1}y, \quad \hat{d} \to z$$

is connected with the initial one (4.23) by an automorphism η of $\mathcal{L}_q$ [8]

$$\eta \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} = \begin{pmatrix} \hat{a} & \omega\hat{b} \\ \omega^{-1}\hat{c} & \hat{d} \end{pmatrix}, \qquad \forall\, \omega \in \mathbb{C}^\times.$$

Nevertheless, being reducible, the module V is obviously indecomposable since matrices $\rho_2(\hat{a})$ and $\rho_2(\hat{d})$ cannot be transformed into diagonal form (unless $x = 0$). Therefore, this module cannot be presented as a direct sum of modules V_ν constructed in Section 3.

So, examining the completeness of the set of V_ν we have to reduce the class of admissible modules to completely reducible ones and reformulate the question in the following way: is any *completely reducible* finite dimensional module over REA (1.1) isomorphic to a direct sum of modules V_ν?

For an arbitrary R-matrix with a finite symmetry rank we have no definite answer to this question. Given the only symmetry rank of R, one has too little information on the concrete structure of the corresponding REA. Perhaps, an analysis of the explicit commutation relations is needed here. The question on irreducibility of modules V_ν themselves is also open in this case.

As for the R-matrix originated from the quantum universal enveloping algebra $U_q(sl_n)$ $(p = n)$ it is highly plausible that the finite direct sums of the modules V_ν do exhaust all finite dimensional completely reducible (non-commutative) representations of REA. The matter is that the matrix elements of the corresponding representations are rational functions in q with nonsingular limit $q \to 1$. At that limit the mREA $\mathcal{SL}_q$ (2.7) tends to the algebra $U(sl_n)$ and all the $\mathcal{SL}_q$ modules V_ν go to the corresponding modules over $U(sl_n)$. In particular the modules V_ν described in Proposition 3 must be irreducible.

To conclude, we shortly summarize the main results and discuss some open problems and perspectives.

For the reflection equation algebra we have constructed the series of finite dimensional non-commutative representations which are parametrized by Young diagrams. The representations exist for any R-matrix satisfying the additional conditions (1.3), (1.6) and (1.9). The corresponding modules V_ν are simple objects of a quasitensor Schur-Weyl category described in detail in [14]. As was pointed out in Section 3, the Grothendiek ring of the Schur-Weyl category for the Hecke R-matrix with the symmetry rank p is isomorphic to that of the category of finite dimensional modules over $U(sl_p)$. Nevertheless, dimensions of the modules and the characters of central elements could drastically differ from each other.

Also, it is worth mentioning some further problems in this approach. First of them is the problem of constructing the representation theory for the REA connected with R-matrices of the

Birman-Murakami-Wenzl type. Examples are given by R-matrices originated from the quantum groups of B, C and D series. The key point here is to develop the adequate technique for the q-analogs of the Young idempotents.

Another interesting problem is the representation theory for the REA with a spectral parameter. This can find a lot of applications to the theory of integrable systems.

Appendix

This is a technical section where some auxiliary formulae of the main text are proved. First, we prove the trace formulae which were used in Proposition 2. The decisive role belongs to the following result.

Lemma 4 *Let R be a solution of the Yang-Baxter equation (1.2), satisfying the additional condition (1.6). Then*

$$Tr_{(0)} B_0 R_{01} R_{02}^{-1} = P_{12} B_1, \tag{4.24}$$

where P is the transposition matrix and B is defined in (1.10).

Proof Rewrite the Yang-Baxter equation (1.2) in the equivalent form

$$R_{12} R_{23} R_{12}^{-1} = R_{23}^{-1} R_{12} R_{23}.$$

Using this equation and definition (1.6) of the skew-inverse matrix Ψ we obtain the following relation

$$Tr_{(0)} \Psi_{10} R_{02} R_{03}^{-1} = P_{12} Tr_{(0)} R_{10}^{-1} R_{20} \Psi_{03} P_{23}.$$

Calculate now the trace in the first space. Since $Tr_{(1)} \Psi_{10} = B_0$, we get

$$Tr_{(0)} B_0 R_{02} R_{03}^{-1} = Tr_{(01)} R_{20}^{-1} P_{12} R_{20} \Psi_{03} P_{23} = Tr_{(0)} \Psi_{03} P_{23} = B_3 P_{23} = P_{23} B_2.$$

This result differs from (4.24) only in the notations of the matrix spaces. ■

So, we are ready to prove the trace formulae used in Proposition 2.

i). $\mathcal{T}(n, k-1) \equiv Tr_{(1)} R^{-1}_{(1\to n)} R^{-1}_{(k-1\to 1)} B_1 R_{1\,k+2}$ at $n < k-1$. First of all, one should use (4.3) in order to draw the chain $R^{-1}_{(1\to n)}$ to the right of $R^{-1}_{(k-1\to 1)}$

$$R^{-1}_{(1\to n)} R^{-1}_{(k-1\to 1)} = R^{-1}_{(k-1\to 1)} R^{-1}_{(2\to n+1)}.$$

The chain $R^{-1}_{(2\to n+1)}$ is evidently commute with $B_1 R_{1\,k+2}$, therefore

$$\mathcal{T}(n, k-1) = R^{-1}_{(k-1\to 2)} \left[Tr_{(1)} R_{12}^{-1} B_1 R_{1\,k+2} \right] R^{-1}_{(2\to n+1)}.$$

Using the cyclic property of trace and then relation (4.24), we come to the desired result

$$\mathcal{T}(n, k-1) = R^{-1}_{(k-1\to 2)} P_{2\,k+2} B_{k+2} R^{-1}_{(2\to n+1)}.$$

ii) The calculation of $\mathcal{T}(k-1, k-1)$ is more cumbersome. The main difficulty is that in this case the chains of R-matrices cannot be drawn through each other. As a consequence, it is not so easy to decrease the number of R-matrices with the indices in the first space in order to apply

(4.24). However, with the help of the Hecke condition and the Yang-Baxter equation the product of R-matrix chains contained in $\mathcal{T}(k-1,k-1)$ can be transformed as follows

$$R^{-1}_{(1\to k-1)}R^{-1}_{(k-1\to 1)} = I_{12\dots k} - \lambda R_1^{-1} - \lambda\sum_{n=2}^{k-1} R^{-1}_{(n\to 2)}R_1^{-1}R^{-1}_{(2\to n)}.$$

The terms in the right hand side contain at most one R matrix with indices in the first space and hence, upon multiplying by $B_1 R_{1\,k+2}$, we can calculate $Tr_{(1)}$ with the help of (4.24). As a result we get the formula which was used in the proof of Proposition 2

$$\mathcal{T}(k-1,k-1) = I_{12\dots k} - \lambda P_{2\,k+2}B_{k+2} - \lambda\sum_{n=2}^{k-1} R^{-1}_{(n\to 2)}P_{2\,k+2}B_{k+2}R^{-1}_{(2\to n)}.$$

iii) At last, relation (4.22) is a direct consequence of (4.24). Indeed, multiply (4.24) by Ψ_{13} from the right and take the trace in the first space. Due to definition (1.6) of the matrix Ψ we find

$$Tr_{(0)}B_0P_{03}R^{-1}_{02} = Tr_{(1)}P_{12}B_1\Psi_{13} = B_2\Psi_{23}\,Tr_{(1)}P_{12} = B_2\Psi_{23}.$$

On the other hand, due to the cyclic property of trace

$$Tr_{(0)}B_0P_{03}R^{-1}_{02} = Tr_{(0)}P_{03}R^{-1}_{02}B_0 = R^{-1}_{32}B_3\,Tr_{(0)}P_{03} = R^{-1}_{32}B_3.$$

Therefore

$$B_2\Psi_{23} = R^{-1}_{32}B_3.$$

Multiplying this by C_3 from the right and using (1.11) we come to the relation

$$q^{2p}B_2\Psi_{23}C_3 = R^{-1}_{32}.$$

Actually this is equivalent to (4.22), since basing on (3.8) one can easily show that

$$B_2\Psi_{23}C_3 = C_3\Psi_{23}B_2.$$

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